

Splitting between quadrupole modes of dilute quantum gas in a two-dimensional anisotropic trap

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Abstract. We consider quadrupole excitations of quasi-two-dimensional interacting quantum gas in an anisotropic harmonic oscillator potential at zero temperature. Using the time-dependent variational approach, we calculate a few low-lying collective excitation frequencies of a two-dimensional anisotropic Bose gas. Within the energy weighted sum-rule approach, we derive a general dispersion relation of two quadrupole excitations of a two-dimensional deformed trapped quantum gas. This dispersion relation is valid for both statistics. We show that the quadrupole excitation frequencies obtained from both methods are exactly the same. Using this general dispersion relation, we also calculate the quadrupole frequencies of a two-dimensional unpolarized Fermi gas in an anisotropic trap. For both cases, we obtain analytic expressions for the quadrupole frequencies and the splitting between them for arbitrary value of trap deformation. This splitting decreases with increasing interaction strength for both statistics. For a two-dimensional anisotropic Fermi gas, the two quadrupole frequencies and the splitting between them become independent of the particle number within the Thomas-Fermi approach.

PACS. 03.75.Fi Phase coherent atomic ensembles; quantum condensation phenomena – 05.30.Jp Boson systems

1 Introduction

In a series of experiments, Bose-Einstein condensates (BEC) have been produced by cooling a vapor of alkali atoms to a temperature of a few nanokelvin [1]. This system opens up interesting perspectives in the field of many body physics. There has been much progress in the theoretical understanding of this system [2]. In particular, the low-energy collective excitation spectrum of a Bose condensed dilute gas in a trap has been discussed analytically by Stringari [3] using a hydrodynamic approximation and also by sum-rule approach. A few low-lying excitations have also been calculated analytically by using time-dependent variational approach [4]. The low-energy excitation spectrum obtained by using time-dependent Gaussian variational ansatz exactly coincides with the hydrodynamic results in the limit of large particle number N . Also a similar type of scaling ansatz has been used to describe the time evolution of the condensate in the large N limit [5–7]. Experimentally the low-lying collective excitation frequencies of a condensate have been measured both at zero temperature [8] and at finite temperature [9]. These observed values of the collective oscillation frequencies are in agreement with theoretical results at zero temperature.

After the discovery of BEC in alkali atomic gas, the behaviour of trapped Fermi gas is also in focus. It is also possible to trap the Fermionic atoms at very low temperature, where the quantum effects can be observed. There has been experimental progress towards cooling a Fermi gas into the degenerate regime ($T < T_F$) [10]. Several authors have studied the thermodynamic properties [11, 12], collective excitation frequencies in the normal phase [13–15] as well as in the superfluid phase [16, 17] of a three-dimensional trapped Fermi gas.

The reduction in dimension of a quantum system is the subject of extensive studies in trapped Bose systems [18–22] as well as trapped Fermi systems [23]. With present technology one can freeze the motion of the trapped particles in one direction to create a quasi two-dimensional quantum gas by tuning the frequency in the z -direction. Recently, the lower dimensional BEC has been realized [24, 25].

In a deformed quantum system, angular momentum is not a good quantum number and the angular momentum states mix with each other. Also in the presence of a small deformation of trap, the states with angular momentum quantum number $+l$ and $-l$ split. The partition function of a deformed two-dimensional harmonic oscillator is exactly the same as the partition function of a rotating harmonic oscillator, where the rotation frequency

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Ω is related to the trap anisotropy, $\Omega = (\omega_y - \omega_x)/2$. Here ω_x and ω_y are the oscillator frequencies of the deformed oscillator. From this analogy we find that the splitting occurs between the $+l$ and $-l$ angular momentum states and it is proportional to the trap anisotropy. Similarly for a quantum gas in a deformed trap, the degenerate multipole modes with angular momentum quantum number $+l$ and $-l$ also split. The dipole excitation frequencies of a quantum gas in deformed harmonic trap are the trap frequencies along x and y -directions, ω_x and ω_y . The frequencies and the splitting of the dipole mode neither depends on the statistics of the trapped gas nor on the interaction between the particles. In circular symmetric trap two quadrupole frequencies are degenerate and the degeneracy is lifted up by elliptic trap deformation. In the present work we consider quadrupole modes of quantum gas in a quasi two-dimensional anisotropic trap. We calculate the frequencies of quadrupole like modes for both bosons and fermions in a deformed harmonic trap analytically. For both statistics, we study the effect of interaction on the splitting between the quadrupole modes for arbitrary deformation of the trap. First we consider the case of a two-dimensional deformed trapped Bose gas at zero temperature. We analyze the nature of small oscillations of the confined gas within the time-dependent variational method. In case of deformed trap the quadrupole mode couples with the monopole mode whereas the scissors mode remains decoupled. This is due to fact that the Hamiltonian is invariant under reflection. We calculate the frequencies of the mixed type of modes using variational technique. Next we construct the most general type of excitation operator which couples the quadrupole and monopole mode. Then we consider only the minimum energy excitation in sum rule approach. The excitation energy obtained from the sum-rule approach exactly matches with that obtained from the time-dependent variational technique even for an arbitrary number of particles. In the case of a symmetric trap, these modes can be identified as a monopole mode and a quadrupole mode. There is a possibility of experimental verification of our results since the quasi two-dimensional Bose condensed has been realized in MIT [24].

This sum-rule method allows us to calculate the monopole mode and the quadrupole modes of a Fermi system in a deformed trap where the equation of motion technique does not hold. These modes can be observed in nano structures like quantum dots. When the two-body interaction is long ranged, the sum rule method can be generalised.

There has been no systematic theoretical study on the collective excitations of a two-dimensional deformed trapped quantum gas at zero temperature. The purpose of this paper is to give an analytic description of the quadrupole excitation frequencies of a two-dimensional deformed trapped quantum gas at zero temperature and to calculate the splitting of the quadrupole modes for an arbitrary deformation of the trap.

The paper is organised as follows. In Section 2, we model the quasi-two-dimensional trapped Bose system.

Using the time-dependent variational method we calculate the monopole and quadrupole excitation frequencies of a two-dimensional deformed trapped interacting Bose gas. In Section 3, using the sum-rule approach we derive a general dispersion relation for the quadrupole excitation frequencies of a two-dimensional trapped quantum system interacting through the two-body potential. This relation is valid for both Fermi and Bose statistics. We apply this general dispersion relation to calculate the same excitation frequencies of a trapped Bose system. We show that the quadrupole excitation frequencies obtained from both methods are exactly the same. In Section 4, we consider trapped unpolarized fermions and apply the dispersion relation obtained from sum-rule approach to calculate the frequencies of quadrupole modes. In Section 4, we present the summary and conclusions of our work.

2 Collective low-energy excitation frequencies of a two-dimensional deformed trapped Bose gas

In BEC experiments, the trap potential can be approximated by an effective three-dimensional harmonic oscillator potential, with tunable trap frequencies ω_z in the axial (z) direction and ω_x, ω_y in the transverse (x - y) plane. The alkali-metal vapors used in experiment are very dilute and the interparticle interaction is well described by the short range pseudopotential and the interaction strength is determined by s -wave scattering length a . Here we consider the case when the interparticle interaction is strongly repulsive. The Gross-Pitaevskii (GP) [26] energy functional of the trapped boson of mass m is given by,

$$E[\psi] = \int d^2r dz \left[\frac{\hbar^2}{2m} |\nabla\psi(\mathbf{r}, z)|^2 + \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) |\psi(\mathbf{r}, z)|^2 + \frac{g}{2} \int d^2r' dz' \delta^2(\mathbf{r} - \mathbf{r}') \delta(z - z') |\psi(\mathbf{r}', z')|^2 |\psi(\mathbf{r}, z)|^2 \right], \quad (1)$$

where, $g = 4\pi a \hbar^2 / m$, \mathbf{r} is the position vector in x - y plane and $\psi(\mathbf{r}, z)$ is the condensate wave function.

It has been shown by Baganato *et al.* [27] that for an ideal two-dimensional Bose gas under harmonic trap, a macroscopic occupation of the ground state can exist at temperature $T < T_c = \sqrt{N/\zeta(2)} \hbar\omega/k_B$. With present technology it is possible to freeze the motion of the trapped particles in one direction to create a quasi-two-dimensional Bose gas. In the frozen direction the particles execute zero point motion. To achieve this quasi-two-dimensional system, the frequency in the frozen direction should be much larger than the frequency in the x - y plane and the mean interactions between the particles. Alternatively, the trap frequencies are such that $\hbar\omega_z \gg \mu \geq \hbar\omega_0$ and $k_B T \ll \hbar\omega_z$, where μ is the chemical potential of the two-dimensional Bose gas.

For a quasi two-dimensional system we may assume that the wave function in the z -direction is separable and is given by,

$$\psi(z) = \frac{1}{(\sqrt{\pi}a_z)^{1/2}} e^{-\frac{z^2}{2a_z^2}}, \quad (2)$$

where $a_z = \sqrt{\hbar/m\omega_z}$ is the oscillator length in the z -direction. Now we integrate out the z -component in the three-dimensional GP energy functional, then we get the effective energy functional in two dimensions:

$$E - \frac{\hbar\omega_z N}{2} = \int d^2r \left[\frac{\hbar^2}{2m} |\nabla\psi(\mathbf{r})|^2 + \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2) |\psi(\mathbf{r})|^2 + \frac{g_2}{2} \int d^2r' \delta^2(\mathbf{r} - \mathbf{r}') |\psi(\mathbf{r}')|^2 |\psi(\mathbf{r})|^2 \right], \quad (3)$$

where $g_2 = 2\sqrt{2\pi}\hbar\omega_z a_z a$ is the effective coupling strength in two dimensions, a is the s -wave scattering length in three dimensions and N is the total number of particles in the condensate. The same effective coupling constant is obtained in reference [19]. The effective interaction in two dimensions is given by,

$$V_1 = g_2 \delta^2(\mathbf{r} - \mathbf{r}'). \quad (4)$$

The chemical potential of quasi two-dimensional Bose condensed state is $\mu = \hbar\omega_0 \sqrt{\sqrt{2\pi}Na/a_z}$. Recently, the two-dimensional Bose condensed state has been realized in MIT [24]. In this experimental set up, they have loaded $N \sim 10^4$ number of sodium atoms in a trap with trap frequencies $\omega_z/2\pi = 790$ Hz, $\omega_0/2\pi = \sqrt{\omega_x\omega_y} \sim 20$ Hz and $a = 2.75$ nm. One can easily calculate the chemical potential $\mu \sim 0.19\hbar\omega_0\sqrt{N}$ which satisfies the above mentioned inequality condition to be a quasi two-dimensional Bose system.

In two dimensions, the equation of motion of the condensate wave function is described by the Gross-Pitaevskii equation,

$$i\hbar \frac{\partial\psi(\mathbf{r})}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g_2 |\psi(\mathbf{r})|^2 \right] \psi(\mathbf{r}), \quad (5)$$

where $V(\mathbf{r}) = m(\omega_x^2 x^2 + \omega_y^2 y^2)/2$ is the deformed trap potential in two dimensions. The normalization condition for ψ is $\int d^2r |\psi|^2 = N$. N is the number of particles in the condensate. One can write down the Lagrangian density corresponding to this system as follows:

$$\mathcal{L} = \frac{i\hbar}{2} \left(\psi \frac{\partial\psi^*}{\partial t} - \psi^* \frac{\partial\psi}{\partial t} \right) + \left(\frac{\hbar^2}{2m} |\nabla\psi|^2 + V(\mathbf{r}) |\psi|^2 + \frac{g_2}{2} |\psi|^4 \right), \quad (6)$$

where $*$ denotes complex conjugation. One can get the non-linear Schrödinger equation (5) by minimizing the action related to the above Lagrangian density (6). In order

to obtain the evolution of the condensate we assume the most general Gaussian wave function,

$$\psi(X, Y, t) = C(t) e^{-\frac{1}{2}[\alpha(t)X^2 + \beta(t)Y^2 + \gamma(t)XY]}, \quad (7)$$

where $C(t)$ is the normalization constant. X and Y are the dimensionless variables, $X = x/a_0$, $Y = y/a_0$ where $a_0 = \sqrt{\hbar/m\omega_0}$ is the oscillator length and $\omega_0 = \sqrt{\omega_x\omega_y}$ is the mean frequency. Further, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ and $\gamma = \gamma_1 + i\gamma_2$ are the time dependent dimensionless complex variational parameters. The α_1 and β_1 are inverse square of the condensate widths in x and y -direction respectively. The square of the normalization constant is $|C(t)|^2 = N\sqrt{D}/\pi a_0^2$, where $D = \alpha_1\beta_1 - \gamma_1^2$. The Gaussian ansatz (Eq. (7)) for the order parameter can also be generalized to three-dimensional anisotropic trapped Bose system to study various scissors modes.

The Gaussian variational ansatz becomes an exact ground state in the non interacting limit and in the presence of repulsive interaction it gives rise to spreading of the condensate wave function. To describe the quadrupoles and monopole oscillation, we consider the most general time dependent quadratic exponent of the variational ansatz.

We obtain the effective Lagrangian L by substituting equation (7) into equation (6) and integrating the Lagrangian density over the space co-ordinates,

$$\frac{L}{N\hbar\omega_0} = \frac{1}{4D} \left[-(\beta_1\dot{\alpha}_2 + \alpha_1\dot{\beta}_2 - 2\gamma_1\dot{\gamma}_2) + (\alpha_1 + \beta_1)D + (\alpha_2^2 + \gamma_2^2)\beta_1 + (\beta_2^2 + \gamma_2^2)\alpha_1 - 2(\alpha_2 + \beta_2)\gamma_1\gamma_2 + \lambda\beta_1 + \frac{\alpha_1}{\lambda} + PD^{3/2} \right], \quad (8)$$

where λ is the asymmetric ratio, $\lambda = \omega_x/\omega_y$ and $P = \sqrt{2/\pi} 2N(a/a_z)$.

The variational energy of the static condensate at equilibrium is given in terms of the equilibrium values of the inverse square width of the condensate along x and y -directions,

$$\frac{E}{N\hbar\omega_0} = \frac{1}{4} \left[(\alpha_{10} + \beta_{10}) + \left(\frac{\lambda}{\alpha_{10}} + \frac{1}{\lambda\beta_{10}} \right) + P\sqrt{\alpha_{10}\beta_{10}} \right]. \quad (9)$$

One can get the equilibrium value of the variational parameters, α_{10} and β_{10} by minimizing the energy with respect to the variational parameters,

$$\alpha_{10}^2 = \lambda - \frac{P}{2}\alpha_{10}\sqrt{\alpha_{10}\beta_{10}}, \quad (10)$$

$$\beta_{10}^2 = \frac{1}{\lambda} - \frac{P}{2}\beta_{10}\sqrt{\beta_{10}\alpha_{10}}. \quad (11)$$

From the above two relations, we obtain

$$\eta^4 + \frac{P\eta^3}{2} - \frac{P\eta}{2\lambda^2} - \frac{1}{\lambda^2} = 0, \quad (12)$$

where η is the ratio of the condensate widths in the x and y -direction, $\eta = \sqrt{\beta_{10}/\alpha_{10}}$. From equation (12) one

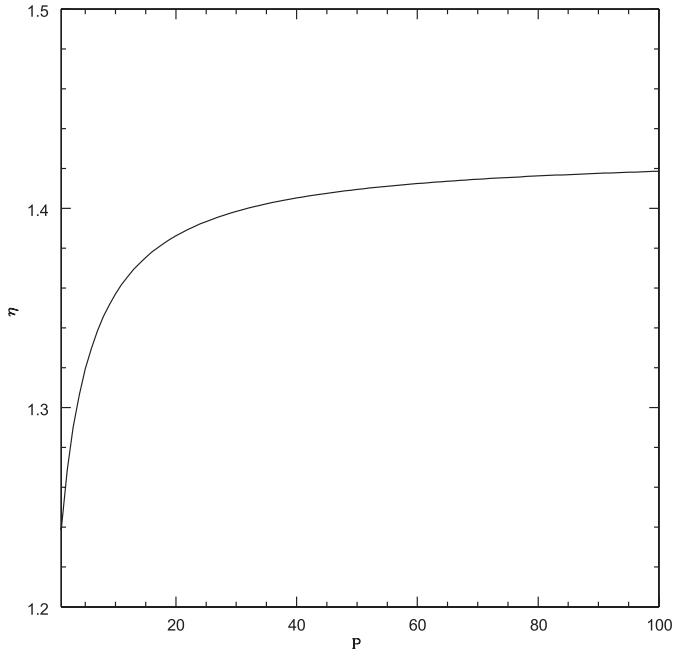


Fig. 1. The variation of the ratio of the widths of the condensate, η as a function of the dimensionless effective interaction strength P , for the fixed ratio of trap frequencies $\lambda = 0.7$.

can say how η changes with the number of atoms N and the coupling constant g_2 . The variation of η with the dimensionless effective interaction strength P is shown in Figure 1. The ratio between the widths of the condensate η varies from $1/\sqrt{\lambda}$ to $1/\lambda$, as the interaction strength increases from zero to large value (Thomas-Fermi limit). In the Thomas-Fermi limit, the equilibrium values of the parameters α_1 and β_1 are,

$$\alpha_{10} = \lambda \sqrt{\frac{2}{P}}, \quad \beta_{10} = \frac{1}{\lambda} \sqrt{\frac{2}{P}}. \quad (13)$$

In this limit, the energy per particle is $E/N = \hbar\omega_0 \sqrt{P/2}$. In the non-interacting limit, $\alpha_{10}^2 = \lambda$ and $\beta_{10}^2 = 1/\lambda$. The energy per particle is $E/N = \hbar\omega_0(\sqrt{\lambda} + \sqrt{1/\lambda})/2$.

We are interested in the low-energy excitations of a Bose system. The low-energy excitations of the condensate correspond to the small oscillations of the cloud around the equilibrium configuration. Therefore we expand the time dependent variational parameters around the equilibrium points in the following way, $\alpha_1 = \alpha_{10} + \delta\alpha_1$, $\beta_1 = \beta_{10} + \delta\beta_1$ and $\alpha_2 = \delta\alpha_2$, $\beta_2 = \delta\beta_2$, $\gamma_1 = \delta\gamma_1$, $\gamma_{10} = 0$ and $\gamma_2 = \delta\gamma_2$.

Using the Euler-Lagrange equation, the time evolution of the inverse square of the width around the equilibrium

points are given by,

$$\delta\ddot{\alpha}_1 + \lambda \frac{(8 + 3P\eta)}{(2 + P\eta)} \delta\alpha_1 + \frac{P\lambda\eta}{(2 + P\eta)} \delta\beta_1 = 0, \quad (14)$$

$$\delta\ddot{\beta}_1 + \frac{P}{\lambda(2\eta + P)} \delta\alpha_1 + \frac{(8\eta + 3P)}{\lambda(P + 2\eta)} \delta\beta_1 = 0, \quad (15)$$

$$\delta\ddot{\gamma}_1 + \left[\frac{4\lambda\eta^2}{(2 + P\eta)} + \left(\lambda + \frac{1}{\lambda} \right) \right] \delta\gamma_1 = 0. \quad (16)$$

From the above equations we can see that the modes corresponding to the fluctuations of the average of x^2 and y^2 are coupled, but the mode associated with the fluctuation of the average value of xy is decoupled. The equations (14, 15) are coupled equations of the modes α_1 and β_1 , where the mode xy is decoupled. This is due to the fact that the Hamiltonian is invariant under reflection, $x \rightarrow -x$ and $y \rightarrow y$ or $x \rightarrow x$ and $y \rightarrow -y$, and the modes which are odd or even under this operation separate out.

Now we look for time dependent solutions of $e^{i\omega t}$ type, we obtain from equations (14, 15),

$$\frac{\omega_{\pm}^2}{\omega_0^2} = \frac{\lambda(8 + 3P\eta)}{2(2 + P\eta)} + \frac{1}{2\lambda} \frac{(8\eta + 3P)}{(2\eta + P)} \pm \sqrt{\left[\frac{\lambda(8 + 3P\eta)}{2(2 + P\eta)} - \frac{(8\eta + 3P)}{2\lambda(2\eta + P)} \right]^2 + \left(\frac{P\lambda\eta^2}{2 + P\eta} \right)^2}. \quad (17)$$

For an isotropic trap, $\omega_+ = 2\omega_0$ and $\omega_-^2 = [\omega_0^2(8 + 2P)]/(2 + P)$. For large N limit, $\omega_- = \sqrt{2}\omega_0$. So ω_+ and ω_- may be identified as the monopole mode frequency and quadrupole mode frequency respectively. The monopole mode is coupled with the quadrupole mode in an anisotropic trap. However, the monopole mode frequency in an isotropic trap is independent of the interaction strength of the two-body potential and the number of particles in the condensate state. This is due to the underlying $SO(2,1)$ symmetry in the Hamiltonian [20,21].

From equation (16) we obtain,

$$\frac{\omega_s^2}{\omega_0^2} = \frac{4\lambda\eta^2}{(2 + P\eta)} + \left(\lambda + \frac{1}{\lambda} \right). \quad (18)$$

In an isotropic trap, ω_s becomes ω_- . In the non-interacting limit, $\omega_s = \omega_x + \omega_y$. In the Thomas-Fermi limit, equation (18) reduces to $\omega_s = \sqrt{\omega_x^2 + \omega_y^2}$. In an isotropic trap this mode corresponds to the quadrupole excitation. This excitation is also known as scissors mode [28], and this oscillation has been observed experimentally [29].

3 Sum-rules and collective excitations

In this section, we study the quadrupole excitations of a two-dimensional deformed trapped quantum gas at zero temperature within the sum rule approach. In the collisionless regime the collective excitation frequencies of a confined gas are well described by the sum rule method.

The collective excitation of any system is usually probed by applying external fields. Given an excitation operator F , many useful quantities of the excited system can be calculated from the so-called strength function [30],

$$S_{\pm}(E) = \sum_n |\langle n|F_{\pm}|0\rangle|^2 \delta(E - E_n), \quad (19)$$

where E_n and $|n\rangle$ are the excitation energy and the excited state respectively, and $F_+ = F$, $F_- = F^\dagger$. Various energy weighted sum rules are derived from the moments of the strength distribution function,

$$m_k^{\pm} = \frac{1}{2} \int E^k (S_+(E) \pm S_-(E)) dE. \quad (20)$$

It is easy to see that, for a given k , the moments may be expressed in terms of the commutators of the excitation operator F with the many body Hamiltonian H . We give below some of the useful energy weighted sum rules,

$$m_0^- = \frac{1}{2} \langle 0|[F^\dagger, F]|0\rangle, \quad (21)$$

$$m_1^+ = \frac{1}{2} \langle 0|[F^\dagger, [H, F]]|0\rangle, \quad (22)$$

$$m_2^- = \frac{1}{2} \langle 0|[J^\dagger, J]|0\rangle, \quad (23)$$

$$m_3^+ = \frac{1}{2} \langle 0|[[F^\dagger, H], [H, [H, F]]]|0\rangle, \quad J = [H, F], \quad (24)$$

where $[,]$ denotes the commutator between corresponding operators. Near the collective excitation frequency the strength distribution becomes sharply peaked, and the collective excitation energy is described by the moments of the strength distribution,

$$\hbar\omega = \sqrt{\frac{m_3^+}{m_1^+}}. \quad (25)$$

Following reference [31], one can derive the above form of collective excitation energy by using the variational principle. Given the many body ground state it is possible to find out the collective excitation energy and the excited state, if one is able to find an operator O^\dagger , which satisfies the following equation of motion:

$$[\hat{H}, O^\dagger] = \hbar\omega_c O^\dagger. \quad (26)$$

The excitation energy is then given by the following expression,

$$\hbar\omega_c = \frac{\langle 0|[O, [\hat{H}, O^\dagger]]|0\rangle}{\langle 0|O, O^\dagger|0\rangle}. \quad (27)$$

We may now take the variational ansatz for O^\dagger as, $O^\dagger = F + bJ$ with the variational parameter b . By minimizing the energy with respect to the variational parameter, we obtain the collective excitation energy as $E_c = \sqrt{m_3^+/m_1^+}$, which is same as equation (25).

Similarly, we construct the most general excitation operator $F = x^2 + by^2$ when monopole and quadrupole modes are coupled. b is a variational parameter. In symmetric trap potential, if $b = 1$, F is monopole mode and if $b = -1$, F is the quadrupole mode. In the same way we can calculate the lowest energy excitation in this particular sector of excitations variationally. The lowest energy mode turns out to be the quadrupole mode.

Calculating the moments m_1 and m_3 by taking the excitation operator with the Hamiltonian, we obtain,

$$E_{\text{coll}}^2 = \frac{4\hbar^2}{m} \frac{E_x}{\langle x^2 \rangle} \frac{(1 + Ab^2 + Bb)}{(1 + Cb^2)}, \quad (28)$$

where $A = E_y/E_x$, $B = E_{\text{int}}/2E_x$, $C = \langle y^2 \rangle / \langle x^2 \rangle$ and $E_x = \langle T_x \rangle + \langle V_x \rangle + \langle E_{\text{int}} \rangle / 4$, $E_y = \langle T_y \rangle + \langle V_y \rangle + \langle E_{\text{int}} \rangle / 4$. Here $\langle \rangle$ denotes the expectation value of the corresponding operators in the ground state and T_x, V_x and T_y, V_y represents the kinetic energy and potential energy along x and y coordinates respectively. The interaction energy is given by $E_{\text{int}} = (g_2/2) \int |\psi|^4 d^2r$. Now we minimize this collective energy with respect to the variational parameter β . The value of β for which the collective energy is minimum, is given by

$$b_0 = \frac{-2(C - A) \pm \sqrt{4(C - A)^2 + 4B^2C}}{2BC}. \quad (29)$$

It can be easily shown that for an isotropic trap, $b_0 = \pm 1$. So we have identified that the monopole mode can be excited by the operator $F = x^2 + b_0y^2$ and quadrupole mode can be generated by $F = x^2 - b_0y^2$. Inserting b_0 into equation (28), we obtain the following collective oscillation frequencies:

$$\omega_{\pm}^2 = \frac{2}{m} \left[\left(\frac{E_y}{\langle y^2 \rangle} + \frac{E_x}{\langle x^2 \rangle} \right) \pm \sqrt{\left(\frac{E_y}{\langle y^2 \rangle} - \frac{E_x}{\langle x^2 \rangle} \right)^2 + \frac{E_{\text{int}}^2}{4\langle x^2 \rangle \langle y^2 \rangle}} \right]. \quad (30)$$

Using the variational wave function of the ground state in deformed trap, one can easily get the excitation frequencies. The lowest energy excitation frequency in this sector is

$$\frac{\omega_-^2}{\omega_0^2} = \frac{\lambda(8 + 3P\eta)}{2(2 + P\eta)} + \frac{1}{2\lambda} \frac{(8\eta + 3P)}{(2\eta + P)} - \sqrt{\left[\frac{\lambda(8 + 3P\eta)}{2(2 + P\eta)} - \frac{(8\eta + 3P)}{2\lambda(2\eta + P)} \right]^2 + \left(\frac{P\lambda\eta^2}{2 + P\eta} \right)^2}. \quad (31)$$

It can be identified as quadrupole mode since in an isotropic trap, its excitation frequency exactly matches with the quadrupole mode frequency. The above expression for the excitation frequency equation (31) is exactly same as the mode frequency ω_- in equation (17).

The higher energy excitation exactly matches within the monopole mode, although it is not the local minimum

of the energy, equation (28). The dispersion relation for this monopole mode frequency is:

$$\frac{\omega_{\mp}^2}{\omega_0^2} = \frac{\lambda(8+3P\eta)}{2(2+P\eta)} + \frac{1}{2\lambda} \frac{(8\eta+3P)}{(2\eta+P)} + \sqrt{\left[\frac{\lambda(8+3P\eta)}{2(2+P\eta)} - \frac{(8\eta+3P)}{2\lambda(2\eta+P)} \right]^2 + \left(\frac{P\lambda\eta^2}{2+P\eta} \right)^2}. \quad (32)$$

For another quadrupole mode, the excitation operator is $F = xy$. Using the commutation relation, we obtain,

$$m_1 = \frac{\hbar^2}{2m} \langle (x^2 + y^2) \rangle, \quad (33)$$

$$m_3 = m_3(T) + m_3(V) + m_3(ee), \quad (34)$$

where,

$$m_3(T) = \frac{\hbar^4}{m^3} \langle (p_x^2 + p_y^2) \rangle, \quad (35)$$

$$m_3(V) = \frac{\hbar^4}{2m} (\omega_x^2 + \omega_y^2) \langle (x^2 + y^2) \rangle, \quad (36)$$

$$m_3(ee) = \frac{g_2 \hbar^4}{2m^2} \left[\int d^2r \rho(r) \left(y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right)^2 \rho(r) + \int d^2r \left(x \frac{\partial \rho}{\partial y} + y \frac{\partial \rho}{\partial x} \right)^2 \right]. \quad (37)$$

Using the variational wave function, we can get all these moments. In this case $m_3(ee)$ exactly vanishes. So the frequency for the quadrupole mode is,

$$\frac{\omega_s^2}{\omega_0^2} = \frac{4\lambda\eta^2}{(2+P\eta)} + \left(\lambda + \frac{1}{\lambda} \right). \quad (38)$$

This expression for the quadrupole frequency is also the same as equation (18). So ω_- in equation (31) and ω_s in equation (38) shows the splitting occurs between two quadrupole modes in a two-dimensional deformed trapped Bose gas and the dependence of the splitting on the interaction strength and trap anisotropy can be analyzed from the analytical expressions. For an isotropic trap, the two quadrupole modes are degenerate. The variation of the splitting between two quadrupole modes $\Delta_b = \omega_s - \omega_-$, of a trapped interacting Bose gas, with the dimensionless interaction parameter P is shown in Figure 2.

We have checked that the sum rule method gives correct results for the excitation frequencies of the two quadrupole modes for a system of interacting bosons in an anisotropic trap. Now we apply this method to calculate the excitation energies of the quadrupole modes of a system of interacting Fermions in a deformed trap.

4 Two-dimensional trapped anisotropic Fermi gas at zero temperature

In this section we discuss the collective oscillation of a two-dimensional deformed trapped unpolarized Fermi gas

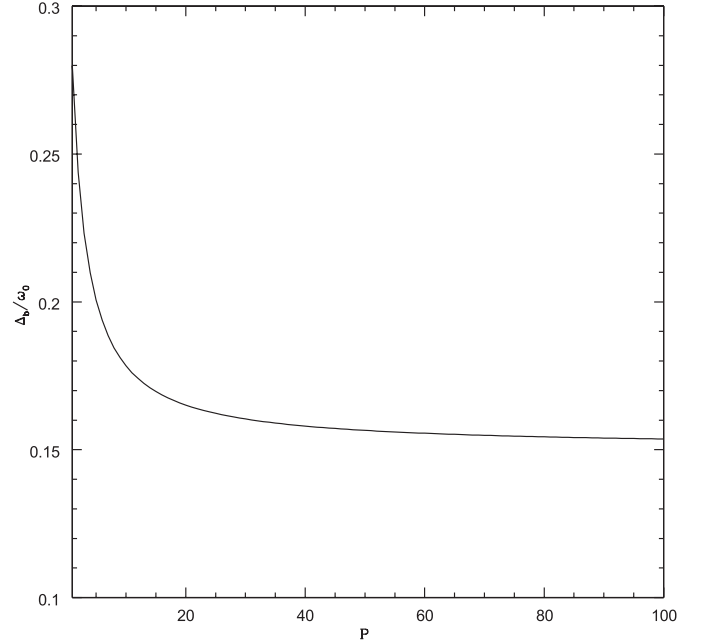


Fig. 2. The difference between the two quadrupole modes of an interacting Bose gas, Δ_b/ω_0 as a function of the dimensionless effective interaction strength P for fixed ratio of trap frequencies $\lambda = 0.7$.

at zero temperature within the sum-rule approach. Using this approach, the collective excitations have been studied in other finite Fermionic systems like atomic nuclei, metal clusters [32] and quantum dots [33].

We consider a two-dimensional deformed trapped unpolarized Fermionic atoms at very low temperature. The two-body interaction of the dilute gas can be described by the short range pseudopotential $V(\mathbf{r} - \mathbf{r}') = g_2 \delta^2(\mathbf{r} - \mathbf{r}')$, where g_2 is the coupling constant and its form is given in Section 2. The Hamiltonian of the trapped Fermionic atoms is given by,

$$H = \sum_i \frac{p_i^2}{2m} + V_{\text{ext}} + g_2 \sum_{i<j} \delta^2(\mathbf{r}_i - \mathbf{r}_j), \quad (39)$$

where the confining potential is

$$V_{\text{ext}} = \frac{1}{2} m \omega_0^2 \left(\lambda x^2 + \frac{y^2}{\lambda} \right). \quad (40)$$

The Thomas-Fermi energy functional of this trapped interacting Fermi system is given by,

$$E[\rho(r)] = \int d^2r \left[\frac{\hbar^2 \pi}{2m} \rho^2(r) + V_{\text{ext}} \rho(r) + \frac{\tilde{g}_2}{2} \rho^2(r) \right], \quad (41)$$

where $\tilde{g}_2 = g_2/2$. Here we assume the density of two spin components are same, $\rho_{\uparrow} = \rho_{\downarrow}$. The interaction energy density $g_2 \rho_{\uparrow} \rho_{\downarrow}$, can be written as $(g_2/4) \rho^2$, where ρ is the total density. By minimizing the energy functional with

respect to density, we obtain,

$$\rho(\mathbf{r}) = \frac{R_F^2}{2K_0\pi a_0^4} \left[1 - \frac{r^2}{R_F^2} \right], \quad r \leq R_F, \quad (42)$$

where $R_F = (4NK_0)^{1/4}a_0$ is the radius of the atomic gas which is determined from the condition $\int d^2r\rho(r) = N$. $K_0 = 1 + \tilde{g}_2m/\pi\hbar^2$ is a dimensionless constant. At very low temperatures, collisions are suppressed due to Fermi statistics and system is in the collisionless regime. We study the collective excitation frequencies in this regime by sum rule approach.

In Section 3 we have derived the expressions for quadrupole excitation frequencies within sum-rule approach. We can use expression equation (30), to calculate one of the quadrupole mode frequencies for Fermi gas also.

We evaluate all the expectation values of the corresponding operators by using the Thomas-Fermi density (42),

$$\frac{E_x}{\langle x^2 \rangle} \pm \frac{E_y}{\langle y^2 \rangle} = \frac{m\omega_0^2(3K_0+1)}{2 \cdot 2K_0} \left(\lambda \pm \frac{1}{\lambda} \right). \quad (43)$$

Using equations (30, 43), we obtain,

$$\frac{\omega_{\pm}^2}{\omega_0^2} = \frac{(3K_0+1)}{2K_0} \left(\lambda + \frac{1}{\lambda} \right) \pm \sqrt{\left[\frac{(3K_0+1)}{2K_0} \left(\lambda - \frac{1}{\lambda} \right) \right]^2 + \left(1 - \frac{1}{K_0} \right)^2}. \quad (44)$$

For an isotropic trap, the monopole mode frequency becomes $\omega_+ = 2\omega_0$.

There is another quadrupole mode for which the excitation operator is $F = xy$. Using the density for the trapped interacting Fermi gas at $T = 0$, we get the following moments:

$$m_1 = \frac{\hbar^3 R_F^6}{48m^2 a_0^6 \omega_0 K_0} \left(\lambda + \frac{1}{\lambda} \right), \quad (45)$$

$$m_3 = \frac{\hbar^5 \omega_0 R_F^6}{12K_0 m^2 a_0^6} \left[\frac{1}{K_0} + \frac{(\lambda + \frac{1}{\lambda})^2}{4} \right]. \quad (46)$$

In this case also, $m_3(ee)$ exactly vanishes. The quadrupole oscillation frequency is given by,

$$\frac{\omega_s}{\omega_0} = \sqrt{\left[\left(\lambda + \frac{1}{\lambda} \right) + \frac{4}{K_0(\lambda + \frac{1}{\lambda})} \right]}. \quad (47)$$

For an isotropic trap, ω_- in equations (44, 47) becomes

$$\omega_s = \sqrt{2}\omega_0 \sqrt{\frac{(1+K_0)}{K_0}}. \quad (48)$$

In equations (44, 47), ω_- exhibits the splitting of the quadrupole modes of a two-dimensional deformed trapped Fermi gas. The scissors mode is also discussed in reference [16] for superfluid Fermi gas.

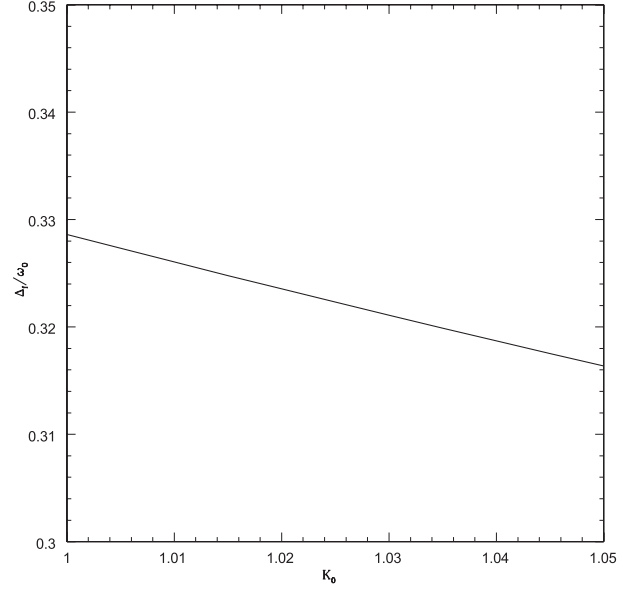


Fig. 3. The difference between the two quadrupole modes of an interacting unpolarized Fermi gas Δ_f/ω_0 , as a function of the dimensionless parameter K_0 for fixed ratio of trap frequencies $\lambda = 0.7$.

The monopole mode frequency of an isotropic trapped interacting Fermi system is $2\omega_0$ which is independent of the interaction strength of the two-body potential and the number of particles. This is because of the presence of $SO(2,1)$ symmetry in the Hamiltonian (Eq. (39)) as discussed in the previous section.

The splitting between two quadrupole modes, $\Delta_f = \omega_s - \omega_-$, of a deformed trapped interacting Fermi gas is shown in Figure 3. The frequencies of these two modes and the splitting between them are independent of the particle number for two-dimensional Fermions within Thomas-Fermi approximation. This splitting decreases almost linearly with increasing interaction strength.

5 Summary and conclusions

In this paper, we have mainly considered two non-degenerate quadrupole modes of a quantum gas in an anisotropic harmonic oscillator potential. We investigated the effect of interaction on the splitting between these quadrupole modes for arbitrary trap deformation. We have calculated a few low-lying collective excitation frequencies of a two-dimensional trapped Bose gas in an anisotropic trap, by using time dependent variational method. We found that one quadrupole mode is coupled with the monopole mode in presence of trap deformation. Another quadrupole mode associated with the fluctuation of the average value of xy (which is also known as scissors mode), is decoupled.

Using the energy weighted sum-rule approach we derived the general dispersion relation of the two quadrupole excitations. Using the same variational wave function for Bosons, we checked that the collective frequencies

obtained from the sum-rule approach are exactly the same as those obtained from the variational method. The main advantage of the sum-rule method is that it can be applied to both trapped Bosons and Fermions to calculate the excitation frequencies in the collisionless regime. This method can be applied for any number of confined particles and also it can be generalised for long range interactions. This energy weighted sum-rule method can be extended for Coulomb interaction to study the quadrupole excitations in a deformed electronic nanostructure like an elliptic quantum dot. The splitting between the quadrupole modes obtained from this method is non perturbative in the trap anisotropy parameter.

We considered a system of two-dimensional spin unpolarised interacting Fermions in an anisotropic harmonic oscillator potential within Thomas-Fermi approximation. Applying the sum-rule technique to this deformed Fermi gas, we obtain two quadrupole excitation frequencies and the splitting between them analytically. For both statistics, the amount of splitting between the quadrupole modes decreases with increasing interaction strength. For a two-dimensional Fermi system the frequencies and the splitting are independent of the particle number. For an isotropic trap, the monopole mode frequency of a Bose gas as well as Fermi gas is the universal frequency $2\omega_0$. This monopole mode frequency is independent of the strength of the two-body interaction potential and the number of particles. This is due to the underlying $SO(2, 1)$ symmetry in the Hamiltonian. Strictly speaking, our all the results are valid when the conditions $\hbar\omega_z \gg \mu \geq \hbar\omega_0$ and $k_B T \ll \hbar\omega_z$ are satisfied.

Recent experimental progress in MIT [24] on quasi two-dimensional Bose condensed shows the possibilities of verification of our results. Above mentioned quadrupole modes are excited in the two-dimensional plane and for simplicity we consider only the two-dimensional trapped gas. This method and the most general Gaussian ansatz for the order parameter can also be extended to three-dimensional anisotropic systems to study the various quadrupole modes. The splitting in these two quadrupole modes may be used to find trap anisotropy. It will be an interesting to study the splitting between the quadrupole modes of an anisotropic quantum system in presence of terms having definite chirality, like magnetic field or rotation.

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